	GOVERNMENT ARTS AND SCIENCE COLLEG, KOVILPATTI – 628 503. (AFFILIATED TO MANONMANIAM SUNDARANAR UNIVERSITY, TIRUNELVELI) DEPARTMENT OF MATHEMATICS STUDY E - MATERIAL CLASS : III B.SC (MATHEMATICS) SEM: V SUBJECT : STATICS (SMMA53)

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SEMESTER – V

CORE PAPER – IX

STATICS (75 Hours) (SMMA53)

Objectives:

- To provide the basic knowledge of equilibrium of a particle
- To develop a working knowledge to handle practical problems

Unit I : Forces acting at a point – parallelogram Law f forces – Triangle of forces – Lami's Theorem – Problems. **16L**

Unit II: Parallel forces and moments – resultant of two parallel forces – resultant of two unlike unequal parallel forces – Varignon's Theorem – Problems. **14L**

Unit III : Equlibrium of three forces acting on a regid body – three coplanar forces theorem – problems. **16L**

Unit IV : Friction – Laws of friction – angle of friction – equilibrium of a particle (i) on a rough inclined plane (ii) under a force parallel to the plane (iii) under any force – problems **15L**

Unit V : Equilibrium of strings – equation of the common catenary – tension at any point – Geometrical properties of common catenary – problems. **14L**

Text Book:

Venkatraman, M.K. - Statics, Agasthiar Publications, Trichy.

Books for Reference:

.S – Statics, Emerald Publishers.
3. Duraipandian, P, Laxmi Duraipandian and Muthamizh Jayapragasam- Mechanics, S.Chand & Company.
1. Narayanan, S-Statics, S.Chand & Company, New Delhi.
2. Viswanatha Naik, K and Kasi, M

UNIT	CONTENTS
I	FORCES ACTING AT A POINT
II	PARALLEL FORCES AND MOMENT
III	EQUILIBRIUM OF THREE FORCES ACTING ON A RIGID BODY
IV	FRICTION
V	EQUILIBRIUM OF STRINGS

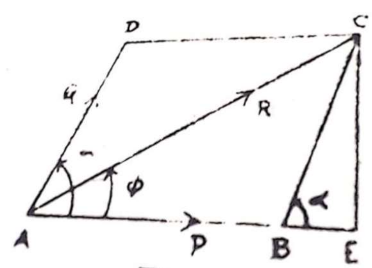
UNIT - I

FORCES ACTING AT A POINT

Definition: If two or more forces F_1,F_2,.... act on a rigid body and if a single force R can be found whose effect on the body is same as that of all the forces F_1,F_2,...F_n,.... then the single orce R is called the resultant of the forces F_1,F_2,.... and the forces F_1,F_2,.... are called the components of the force R.

Parallelogram of forces:
If two forces acting at a point be represented in magnitude and direction, by the sides of a parallelogram drawn from a point, their resultant both in magnitude and direction by the diagonal of the parallelogram drawn through the point.

Analytical expression for the resultant of two forces acting at a point:



Let the two forces P and Q acting at A be represented by AB and AD and let the angle between them be α.
Complete the parallelogram BAD.
Then the diagonal AC will represent the resultant.
Let R be the magnitude of the resultant and let it make an angle φ with P.
Draw CE perpendicular to AB.

From right angle triangle ΔCBE $sin\angle CBE=CEBC$ $ie)sin\alpha=CEQ$
$\Rightarrow CE=Qsin\alpha$ $cos\angle CBE=BEBC$ $ie)cos\alpha=BEQ$
$\Rightarrow BE=Qcos\alpha$ now $R^2=AC^2=AE^2+CE^2=(AB+BE)^2+CE^2=(P+Qcos\alpha)^2+(Qsin\alpha)^2$
$=P^2+Q^2+2PQcos\alpha$ $\therefore R=\sqrt{P^2+Q^2+2PQcos\alpha}$
Also $tan\varphi=CEAE=Qsin\alpha P+Qcos\alpha\alpha$
The above two equations gives the magnitude and direction of the resultant of two forces.

Corollary 1:

If the forces P and Q are at right angles to each other, then α=90 $R=\sqrt{P^2+Q^2}$
And $tan\varphi=QP$
Hence the parallelogram becomes a rectangle.

Corollary 2:

If the two forces are equal, then $R=\sqrt{P^2+P^2+2PPcos\alpha}=\sqrt{2P^2(1+cos\alpha)}$

$=\sqrt{2P^2 2\cos^2\frac{\alpha}{2}} = 2P\cos\frac{\alpha}{2}$

And $\tan\varphi = \frac{P\sin\alpha}{P+P\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} \Rightarrow \varphi = \frac{\alpha}{2}$

Thus the resultant of two equal forces in a direction bisecting the angle between them.

Corollary 3:

Let the magnitudes P and Q of two forces acting at an angle α be given.
Then their resultant R is greatest when cosαis greatest.
The maximum value of cos α is 1.
Therefore the resultant is R=P+Q
In this case the forces acting along the same line and same direction
their resultant R is least when cosαis least.
The minimum value of cos α is -1.
Therefore the resultant is R=P-Q
In this case the forces acting along the same line but opposite direction

Problem 1:

The resultant of two forces P,Q acting on a certain angle is X, and that of P,R acting at the same angle is also X. The resultant of Q,R again acting at the same angle is Y. Prove that $P=(X^2+QR)^{\frac{1}{2}}=\frac{QR(Q+R)}{Q^2+R^2-Y^2}$, prove also that, if P+Q+R=0,Y=X

Solution: Let P and Q act at an angle α

From the given data we have the following equations: $X^2=P^2+Q^2+2PQcos\alpha$ ....(1)

$X^2=P^2+R^2+2PRcos\alpha$ ....(2) $Y^2=Q^2+R^2+2RQcos\alpha$ ....(3)

(1)-(2) gives $0=Q^2-R^2+2pcos\alpha(Q-R)=(Q-R)(Q+R+2P\cos\alpha)$

But $Q\neq R$ $Q+R+2Pcos\alpha=0$

It gives $cos\alpha=-\frac{(Q+R)}{2P}$

Substituting in (1) we get $X^2=P^2+Q^2+2PQ-\frac{(Q+R)}{2P}=P^2+Q^2-Q^2-QR=P^2-QR$

$\Rightarrow P=(X^2+QR)^{\frac{1}{2}}$

Substituting the value of cos α in (3) we get $Y^2=R^2+Q^2+2RQ-\frac{(Q+R)}{2P}=R^2+Q^2-\frac{QR(Q+R)}{P}$

$\frac{QR(Q+R)}{P}=R^2+Q^2-Y^2$ $P=\frac{QR(Q+R)}{Q^2+R^2-Y^2}$

Hence P=$(X^2+QR)^{\frac{1}{2}}=\frac{QR(Q+R)}{Q^2+R^2-Y^2}$

Given P+Q+R=0, then Q+R=-P $\therefore cos\alpha=P2P=12$
Putting this values in (2) and (3) we get $X^2=P^2+Q^2+PQ$ $Y^2=R^2+Q^2+RQ$ $X^2-Y^2=P^2-Q^2+PR-QR$ $=(P-Q)(P+Q+R)=0$
Therefore X=Y

Triangle of forces:

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

Perpendicular triangle of forces:

If three forces acting at a point are such that their magnitude are proportional to the sides of a triangle and their direction are perpendicular to the corresponding sides, all inwards are all outwards, then also the forces will be in equilibrium.

Converse of the triangle of forces:

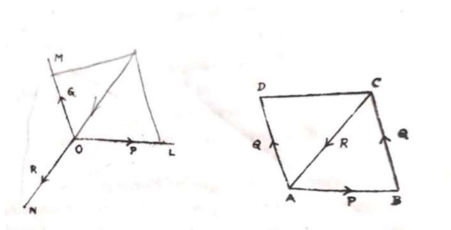
If three forces at a point are in equilibrium, then any triangle drawn so as to have its sides parallel to the direction of the forces shall represent them in magnitude also.

The polygon of forces:

If any number of forces at a point can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.

Lami's theorem:

If three forces acting at a point are in equilibrium, each force is proportional to the sine of the angle between the other two.

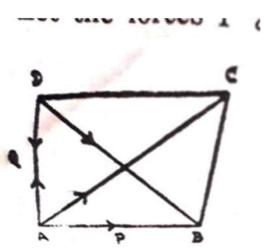


Let P,Q,R be three forces acting one point O.
By triangle of forces, we can prove that the sides of the triangle OAD represent the forces P,Q,R in magnitude and direction.
Applying the sine rule for the triangle OAD $OAsin\angle ODA=ADsin\angle DOA=DOsin\angle OAD$
$\Rightarrow OA\sin(180-\angle MON)=AD\sin(180-\angle NOL)=DO\sin(180-\angle LOM)$
$\Rightarrow OAsin\angle MON=ADsin\angle NOL=DOsin\angle LOM$
$\Rightarrow Psin\angle ODA=Qsin\angle DOA=Rsin\angle OAD$
$\Rightarrow P\sin(Q,R)=Q\sin(P,R)=R\sin(P,Q)$

Problem 1:

Two forces act on a particle. If the sum and difference of the forces are at right angles to each other, show that the forces are of equal magnitude.

Solution:

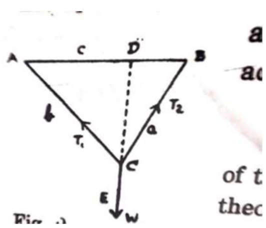


Let the forces P and Q acting at A be represented in magnitude and direction by the lines AB and CD.
Complete the parallelogram BAD.
Then $P+Q=\overline{AB}+\overline{AD}=\overline{AC}$ (using parallelogram law)
Therefore $\overline{AC}$ *is* the sum of the two forces.
$P-Q=\overline{AB}-\overline{AD}=\overline{AB}+\overline{DA}=\overline{DB}$(using triangle law)
$\overline{DB}$is the difference of two forces.
It is given that $\overline{AC}$ and $\overline{DB}$ are at right angles.
Therefore we have the diagonals are at right angles
Hence ABCD must be rhombus.
Therefore AB=AD, ie)P=Q
The forces are equal.

Problem 2:

A and B are two fixed points on a horizontal line at a distance c apart. Two fine light strings AC and BC of lengths b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b(a^2+c^2-b^2):a(b^2+c^2-a^2)$

Solution:



Let T_1 and T_2be the tensions along the strings CA and CB and W, the weight of the mass at C, acting vertically downwards along CE.
Produce EC to meet AB at D.
Since C is at rest under the action of the three forces, we have by the lami’s theorem

$T_1 sin\angle ECB = T_2 sin\angle ECA$

Now $sin\angle ECB = \sin(180-\angle DCB) = sin\angle DCB = \sin(90-\angle ABC) = cos\angle ABC$

$sin\angle ECA = \sin(180-\angle ACD) = sin\angle ACD = \sin(90-\angle BAC) = cos\angle BAC$

Therefore we get $\frac{T_1}{cos\angle ABC} = \frac{T_2}{cos\angle BAC} \Rightarrow \frac{T_1}{cosB} = \frac{T_2}{cosA} \Rightarrow \frac{T_1}{T_2} = \frac{cosB}{cosA}$

$= \frac{\frac{c^2+a^2-b^2}{2ca}}{\frac{c^2+b^2-a^2}{2cb}} = \frac{b(c^2+a^2-b^2)}{a(c^2+b^2-a^2)}$

Therefore the tensions of the strings are in the ratio $b(a^2+c^2-b^2):a(b^2+c^2-a^2)$

Problem 3:

ABC is a given triangle. Forces P,Q,R acting along the lines OA,OB,OC are in equilibrium. Prove that

i i) $P:Q:R = a^2(b^2+c^2-a^2):b^2(c^2+a^2-b^2):c^2(a^2+b^2-c^2)$ if O is the circumcentre of the triangle.

ii ii) $P:Q:R = cos\frac{A}{2}:cos\frac{B}{2}:cos\frac{C}{2}$ if O is the incentre of the triangle.

iii iii) $P:Q:R = a:b:c$ if O is the orthocentre of the triangle.

iv iv) $P:Q:R = OA:OB:OC$ if O is the centroid of the triangle.

Solution:

By lami's theorem, we have $Psin\angle BOC=Qsin\angle COA=Rsin\angle AOB$ ….(1)

i) When O is the circumcentre of the triangle ABC

$\angle BOC=2\angle BAC=2A$

Similarly $\angle COA=2B,\angle AOB=2C$

Therefore (1) gives $Psin2A=Qsin2B=Rsin2C \Rightarrow P2sinAcosA=Q2sinBcosB=R2sinCcosC$

But in triangle ABC, $cosA=(b2+c2-a2)2bc, cosB=(a2+c2-b2)2ac, cosC=(b2+a2-c2)2ba$

Also $sinA=2\Delta bc, sinB=2\Delta ac, sinC=2\Delta ab$

Substitute all of these values we get $P2(b2+c2-a2)2bc2\Delta bc=Q2(a2+c2-b2)2\Delta ac2ac=R2(b2+a2-c2)2ba2\Delta ab$

$\Rightarrow Pb2c2(b2+c2-a2)=Qa2c2(a2+c2-b2)=Rb2a2(b2+a2-c2)$

$\Rightarrow Pa2(b2+c2-a2)=Qb2(a2+c2-b2)=Rc2(b2+a2-c2)$

ii) When O is the incentre of the triangle, OB and OC are the bisectors of $\angle B$ and $\angle C$

$\therefore \angle BOC=180-B2-C2$

$=180-(B2+C2)$ $=180-(90-A2)$ $=90+A2$ *similarly* $\angle COA=90+B2$ *and* $\angle AOB=90+C2$

Therefore (1) becomes $P\sin(90+A2)=Q\sin(90+B2)=R\sin(90+C2) \Rightarrow PcosA2=QcosB2=RcosC2$

iii) Let O be the orthocentre of the triangle

In the above figure AD,BE,CF are altitudes.
Quadrilateral AFOE is cyclic $\therefore \angle FOE+A=180 \Rightarrow \angle FOE=180-A$ $\angle BOC=vertically\ opposite\ of$ $\angle FOE=180-A$
Similarly $\angle COA=180-B$ and $\angle AOB=180-C$
Hence (1) becomes $P\sin(180-A)=Q\sin(180-B)=R\sin(180-C) \Rightarrow PsinA=QsinB=RsinC\ since$ $in\ triangle\ asinA=bsinB=csinC$
Combining the above equations we get
$Pa=Qb=Rc$
iv) When O is the centroid of the triangle,

$\Delta BOC=\Delta COA=\Delta AOB\ and\ each=13\Delta ABC$ $\Delta BOC=12OB.OCsin\angle BOC=13\Delta ABC$
$\therefore sin``\angle BOC=2\Delta ABC3OB.OC$
Similarly $sin\angle COA=2\Delta ABC3OA.OC$ and $sin\angle AOB=2\Delta ABC3OB.OA$
Hence (1) becomes $P3OB.OC2\Delta ABC=Q3OA.OC2\Delta ABC=R3OB.OA2\Delta ABC \Rightarrow P.OB.OC=Q.OA.OC=R.OB.OA$
Problem 4:
Weights W,w,W are attached to points B,C,D respectively of a light string AE where B,C,D divide the string into 4 equal lengths. If the string hangs in the form of 4 concecutive sides of a rectangular octagon with the ends A and E attached to points on the same level, show that $W=(\sqrt{2}+1)w$
Solution:

ABCDE is a part of a regular octagon.
We know that each interior angle of a regular polygon of n sides =$(2n-4n)\times 90$
Putting n=8, we get each interior angle is 135
Let the tensions in the portion Ab,BC,CD,DE be $T1,T2,T3,T4$ respectively. The string BC pulls B towards C and pulls C towards B, the tension being the same throughout its length. This fact is used to denote the forces acting at B,C and D.
In $\Delta BCD,\angle BCD=135$ $\therefore\angle CBD=\angle CDB=452=2212$ $\angle ABD=\angle ABC-\angle CBD=135-2212=11212$
we know that every regular polygon is cyclic.
Therefore A,B,C,D,E lie on the same circle. $\therefore\angle EAB=180-\angle BDE$ $=180-(\angle CDE-\angle BDC)$ $=180-(135-2212)$ $=6712$ $\therefore\angle EAB+\angle ABD=6712+11212=180$ $\therefore AE||BD$
BD also in horizontal.
Let the vertical line through B meet AE at L and the vertical line through C meet BD at M.

Applying Lami's theorem for the forces at B, we get $Wsin\angle ABC=T_2\sin\ (180-\angle ABL)$
$\Rightarrow Wsin135=T_2\sin\angle ABL \Rightarrow Wsin135=T_2\sin22_{12} \Rightarrow T_2=Wsin22_{12}sin135$ ….(1)
Similarly applying lami's theorem for the forces at C, $wsin\angle BCD=T_2\sin\ (180-\angle MCD)$
$\Rightarrow wsin135=T_2\sin\angle MCD \Rightarrow wsin135=T_2\sin(90-22_{12})\Rightarrow wsin135=T_2\cos22_{12}$
$\Rightarrow T_2=wcos22_{12}sin135$ ….(2)
Equating the two equations we get $Wsin22_{12}sin135=wcos22_{12}sin135 \Rightarrow wW=tan2212=\sqrt{}2-1$
$\Rightarrow w=W(\sqrt{}2-1) \Rightarrow W=w\sqrt{}2-1\ W=w(\sqrt{}2+1)$
Problem 5:

A weight is supported on a smooth plane of inclination α by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β and the slope of the string unaltered, the tension of the string is doubled. Prove that $cot\alpha - 2cot\beta = tan\gamma$

Solution:

Fig. 13

P is the position of the weight. The forces acting at P are

i) Itsweight W downwards

ii) ii) The normal reaction R perpendicular to the inclined plane

iii) The tension T along the string at an angle γ to the horizontal

By lami's theorem for the forces at P, $T\sin(180-\alpha)=W\sin(90-(\gamma-\alpha))$ $\Rightarrow T sin\alpha = W\cos(\gamma-\alpha)$

$\therefore T = W sin\alpha \cos(\gamma-\alpha)$

In the second case, the inclination of the plane is β

There is no change in γ

If T_1 is the tension in the string we will have $T_1 = W sin\beta \cos(\gamma-\beta)$

Given that $T_1 = 2T$ $\Rightarrow W sin\beta \cos(\gamma-\beta) = 2W sin\alpha \cos(\gamma-\alpha)$ $\Rightarrow sin\beta \cos(\gamma-\alpha) = 2sin\alpha \cos(\gamma-\beta)$

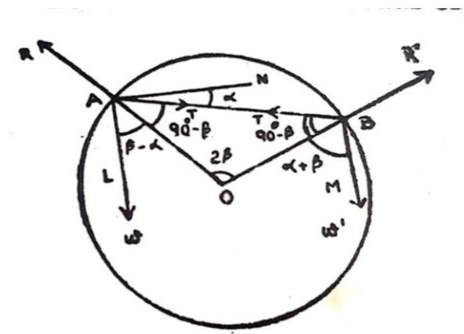
$\Rightarrow sin\beta(cos\gamma cos\alpha + sin\gamma sin\alpha) = 2sin\alpha(cos\gamma cos\beta + sin\gamma sin\beta) \Rightarrow sin\beta cos\gamma cos\alpha + sin\beta sin\gamma sin\alpha = 2sin\alpha cos\beta cos\gamma + 2sin\alpha sin\beta sin\gamma \Rightarrow sin\beta sin\gamma sin\alpha = sin\beta cos\gamma cos\alpha - 2sin\alpha cos\beta cos\gamma$

$\Rightarrow sin\gamma = cos\gamma cos\alpha sin\alpha - 2cos\beta cos\gamma sin\beta \Rightarrow sin\gamma cos\gamma = cos\alpha sin\alpha - 2cos\beta sin\beta$
$\Rightarrow tan\gamma = cot\alpha - 2cot\beta$

Problem 6:

Two beads of weights w and w' can slide on a smooth circular wire in a vertical plane. They are connected by a light string which subtends an angle 2β at the centre of the circle when the beads are in equilibrium on the upper half of the wire.prove that the inclination of the string to the horizontal is given by $tan\alpha = \frac{w \sim w'}{w+w'} tan\beta$

Solution:



Let A and B be the beads of weights w and w' connected by a light string on a circular wire.
In the equilibrium position, $\angle AOB = 2\beta$. O being the centre of the circle. $\therefore \angle OAB = \angle OBA = 90 - \beta$
Let AB make an angle α to the horizontal AN.
AL and BM are the vertical lines through A and B. $\angle OAL = 90 - \angle OAN = 90 - (\angle OAB + \angle NAB)$
$= 90 - (90 - \beta + \alpha) = \beta - \alpha$
Since AL||BM, $\angle AABM + \angle BAL = 180 \therefore \angle ABM = 180 - \angle BAL = 180 - (90 - \alpha) = 90 + \alpha$

$\therefore \angle OBM = \angle ABM - \angle ABO = 90 + \alpha - (90 - \beta) = \alpha + \beta$
The forces acting on the beads w at A are
i) Weight w acting vertically downwards along AL

ii) Normal reaction R due to contact with the wire along the radius OA outwards

iii) Tension T in the string along AB
Similarly the forces acting on the beads w' at B are
i) Weight w' acting vertically downwards along BM

ii) Normal reaction R' due to contact with the wire along the radius OB outwards

iii) Tension T in the string along BA
Apply lami's theorem for the three foces at A $w\sin(180-(90-\beta)) = T\sin(180-(\beta-\alpha))$
$\Rightarrow w cos\beta = T\sin\,(\beta-\alpha)$ ....(1)
Apply lami's theorem for the three forces at B $w'\sin(180-(90-\beta)) = T\sin(180-(\beta+\alpha))$
$\Rightarrow w' cos\beta = T\sin\,(\beta+\alpha)$ ....(2)
Dividing (1) by (2) we have $ww' = \sin\,(\beta+\alpha)\sin(\beta-\alpha)$
Now $w-w'w+w' = \sin(\beta+\alpha)-\sin(\beta-\alpha)\sin(\beta+\alpha)+\sin(\beta-\alpha)$ $= 2cos\beta sin\alpha 2 sin\beta cos\alpha$ $= tan\alpha tan\beta$ *hence*
$tan\alpha = w-w'w+w' tan\beta$

UNIT - II

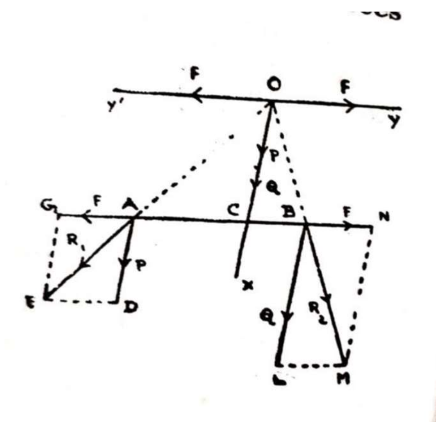
PARALLEL FORCES AND MOMENTS

Definition:

Two parallel forces are said to be like when they act in the same direction

Two parallel forces are said to be unlike when they act in the opposite direction

Resultant of two like parallel forces acting on a rigid body



Let the like parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL. At A and B, introduce two equal and opposite force F of arbitrary magnitude along the line AB and let them be represented be Ag and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at the point A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram ADEG. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.

Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can considered to act at O. At O draw Y'OY||AB and OX||the direction of P and Q.

Resolve R_1 and R_2 at O into their original components.

R_1at O is equal to a force F along OY' and a force P along OX. R_2at O is equal to a force F along OY and a force Q along OX. The two F's at O cancel each other, being equal and opposite.

Hence their resultant is a force P+Q acting along OX.

Thus the magnitude of the resultant of two like parallel forces is their sum. The direction of the resultant is parallel to the components and in the same sense

To find the position of the resultant:
Let OX meet AB at C.
Triangles OAC and AED are similar $\therefore OCAD=ACED \Rightarrow OCP=ACF \Rightarrow F.OC=P.AC$
Triangles OCB and BLM are similar $\therefore OCBL=CBLM \Rightarrow OCQ=CBF \Rightarrow F.OC=Q.CB \; we \; get \; P.AC=Q.CB$
The point C divides AB internally in the inverse ratio of the forces.
Resultant of two unlike parallel forces acting on a rigid body

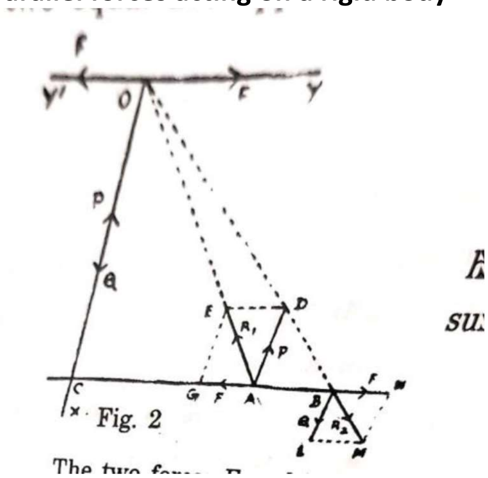


Fig. 2

Let the unlike parallel forces P and Q act at the points A and B of the rigid body respectively and let them be represented by the lines AD and BL with P>Q. At A and B, introduce two equal and opposite force F of arbitrary magnitude along the line AB and let them be represented be AG and BN. These two new forces will balance each other and hence will not affect the resultant of the system.

The two forces F and P acting at the point A can be compounded into a single force R_1 represented by the diagonal AE of the parallelogram ADEG. Similarly the two forces F and Q acting at the point B will have a resultant R_2 represented by the diagonal BM of the parallelogram BLMN.
Produce EA and MB and let them meet at O. The two resultants R_1 and R_2 can considered to act at O. At O draw Y'OY||AB and OX||the direction of P and Q.
Resolve R_1 and R_2 at O into their original components.
R_1at O is equal to a force F along OY' and a force P along OX. R_2at O is equal to a force F along OY and a force Q along OX. The two F's at O cancel each other, being equal and opposite.
Hence their resultant is a force P-Q acting along XO.
Thus the magnitude of the resultant of two unlike parallel forces is their difference. The direction of the resultant is parallel to and in the sense of greater component.

To find the position of the resultant:

Let OX meet AB at C.
Triangles OAC and AED are similar $\therefore OCAD=ACED \Rightarrow OCP=ACF \Rightarrow F.OC=P.CA$
Triangles OCB and BLM are similar $\therefore OCBL=CBLM \Rightarrow OCQ=CBF \Rightarrow F.OC=Q.CB \; we \; get \; P.CA=Q.CB$
The point C divides AB externally in the inverse ratio of the forces.

Condition of equilibrium of three coplanar parallel forces:

Let P,Q,R be three forces parallel in one plane and be in equilibrium. Draw a line to meet the line of action of these forces A,B and C respectively.

If all the three forces are in the same sense, equilibrium will be clearly impossible. Hence two of them must be like and the third R unlike.

The resultant of P and Q is P+Q parallel to P or Q and hence for equilibrium R must be equal and opposite to P+Q.

Therefore R=P+Q and the line of action of P+Q must pass through C.

P.AC=Q.CB $PCB=QAC=P+QCB+AC=P+QAB=RAB$

Hence $PCB=QAC=RAB$

Thus if three parallel forces are in equilibrium, each is proportional to the distance between the other two.

Moment of a force:

When forces act on a particle, the only motion that can occur is a motion of translation. But a force acting on a rigid body may produce either a motion of translation or rotation combined. When there is a motion of translation alone the force is measured by the products of the mass of the particle and the acceleration produced on it by the force. In the case of rotation, the idea of the turning effect or moment of a force is introduced.

The moment of a force about a point is defined to be the product of the force and the perpendicular distance of the point from the line of action of the force.

The moment of a force about a point is zero either

i) The force itself is zero

ii) The line of action of the force passes through the point.

Varigon's theorem:

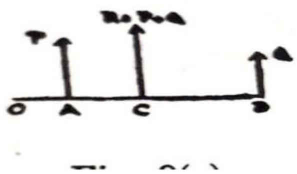
The algebraic sum of the moments of two forces about any point in their plane is equal to the moment of their resultant about that point.

Proof:

To prove this theorem we consider two cases.

Case i)

Let the forces be parallel



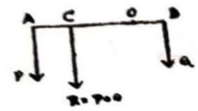
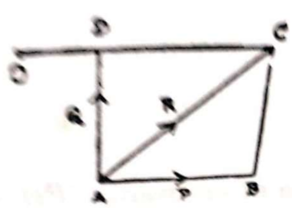


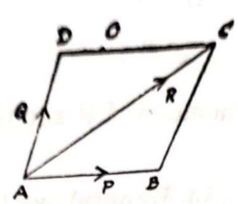
Fig. 9(b)

Let P and Q be two parallel forces and O any point in their plane. Draw AOB perpendicular to the forces to meet their lines of action in A and B.
The resultant of P and Q is a force R acting at C such that P.AC=Q.CB
The algebraic sum of the moments of P and Q about O=P.OA+Q.OB
=P.(OC-AC)+Q(OC+CB)
=(P+Q)OC-P.AC+Q.CB
=(P+Q)OC
=R.OC
=moment of R about O
If O is with in AB, then
The algebraic sum of the moments of P and Q about O=P.OA-Q.OB
=P.(OC+AC)+Q(CB-CO)
=(P+Q)OC+P.AC-Q.CB
=(P+Q)OC
=R.OC
=moment of R about O.
When the parallel forces P and Q are unlike and unequal, the theorem can be proved exactly in the same way.

Case ii):

Let the force meet at a point.





Let the two forces P and Q act at A and let O be any point in their plane. Through O draw a line parallel to the direction of P meeting the line of action of Q at D. Choose the scale of representation such that length AD represents Q in magnitude. On the same scale, let length AB represent P. Complete the parallelogram BAD so that the diagonal AC represent the resultant R of P and Q.
Moment of P,Q,R about O are represented by $2\Delta AOB$,$2\Delta AOD$,$2\Delta AOC$ respectively.
If O lies outside angle BAD and the moments of P and Q are both positive.
The algebraic sum of the moments of P and Q=$2\Delta AOB+2\Delta AOD=2\Delta ACB+2\Delta AOD$ $=2\Delta ADC+2\Delta AOD=2\Delta AOC=moment\ of\ R\ about\ O\ if\ O\ lies\ inside\ the\ angle\ BAD$,$the\ moment\ of\ P\ about\ O\ is\ positive\ while\ that\ of\ Q\ is\ negative$

The algebraic sum of the moments of P and Q=$2\Delta AOB-2\Delta AOD=2\Delta ACB-2\Delta AOD$
$=2\Delta ADC-2\Delta AOD=2\Delta AOC=moment\ of\ R\ about\ O$

Generalised theorem of moments:

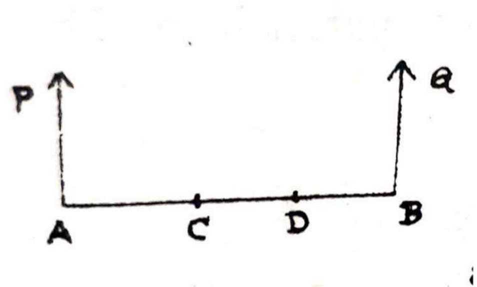
If any number of coplanar forces acting on a rigid body have a resultant, the algebraic sum of their moments about any point is equal to the moment of the resultant about the same point.

Problem 1:

Two like parallel forces P and Q act on a rigid body at A and B respectively:
a) If Q be changed to $\frac{P^2}{Q}$, show that the line of action of the resultant is the same as it would be if the forces were simply interchanged.

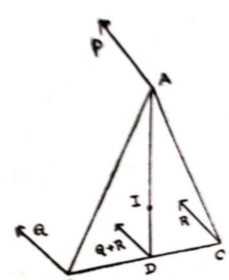
b) If P and Q be interchanged in position, show that the point of application of the resultant will be displaced along AB through a distance d, where $d=\frac{P-Q}{P+Q}.AB$

Solution:



a) Let C be the centre of two parallel forces with P at A and Q at B.
Then P.AC=Q.CB ....(1)
If Q is changed to $\frac{P^2}{Q}$, let D be the new centre of parallel forces.
Then $P.AD=\frac{P^2}{Q}DB$
$\Rightarrow Q.AD=P.DB$....(2)
The above equation shows that D is the centre of two like parallel forces with Q at A and P at B.
b) When the forces P and Q are interchanged in position, D is the new centre of parallel forces.

CD=d
From (2) Q.(AC+CD)=P.(CB-CD) $\Rightarrow Q.AC+Q.d=P.CB-P.d \Rightarrow (Q+P).d=P.CB-Q.AC$
$=P(AB-AC)-Q(AB-CB)=P.AB-P.AC-Q.AB+Q.CB=(P-Q).AB \Rightarrow d=\frac{P-Q}{P+Q}.AB$
Problem 2:
Three like parallel forces, acting at the vertices of a triangle, have magnitudes proportional to the opposite sides. Show that their resultant passes through the incentre of the triangle.
Proof:



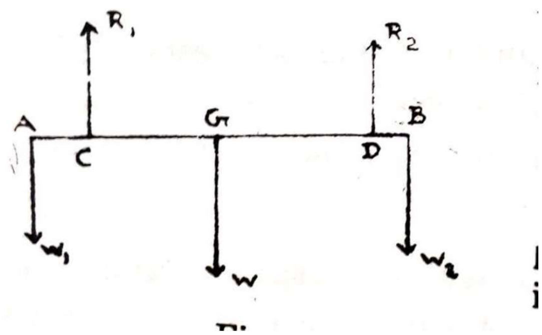
Let like parallel forces P,Q,R act at A,B,C.
It is given that $\frac{P}{a}=\frac{Q}{b}=\frac{R}{c}$ ...(1)
Let the resultant of Q and R meet BC at D.
We know that the magnitude of the resultant is Q+R

Also $BDDC=force\ at\ Cforce\ at\ B=RQ=cb=ABAC$
Therefore AD is the internal bisector of A
We have now to find the resultant of the two like parallel forces, Q+R at D and P at A.
Let this resultant meet AD at I
Then $AIID=force\ at\ Dforce\ at\ A=Q+RP=b+ca$
From above, it is clear that I is the incentre of the triangle.

Problem 3:

A uniform plank of length 2a and weight W is supported horizontally on two vertical propos at a distance b apart. The greatest weight that can be placed at the two ends in succession without upsetting the plank are W_1 and W_1 respectively. Show that $W_1W+W_1+W_2W+W_2=ba$

Solution:



Let AB be the blank placed upon two vertical props at C and D. CD=b. The weight W of the plank acts at G, the midpoints of AB.
AG=GB=a
When the weight W_1 is placed at A, the contact with D is just broken and the upward reaction at D then is zero.
There is upward reaction R_1 at C.
Now, taking moments about C, we have $W_1.AC=W.CG \Rightarrow W_1(AG-CG)=W.CG$
$\Rightarrow W_1AG=(W+W_1)CG \Rightarrow W_1a=(W+W_1)CG \Rightarrow CG=W_1aW+W_1$
When the weight W_2 is attached at B, there is loose contact at C. The reaction at C becomes zero.
There is upward reaction R_2 about D.
Now taking moments about D, we get $W.GD=W_2BD \Rightarrow W.GD=W_2(GB-GD)$
$\Rightarrow GD(W+W_2)=W_2GB=W_2a \Rightarrow GD=W_2aW+W_2$

But $CG+GD=CD=b \Rightarrow W_1aW+W_1+W_2aW+W_2=b \Rightarrow W_1W+W_1+W_2W+W_2=ba$

Problem 4:

The resultant of three forces P,Q,R acting along the slides BC,CA,AB of a triangle ABC passes through the orthocentre. Show that the triangle must be obtuse angled. If $\angle A=120$ *and* $B=C$,*show that* $Q+R=p\sqrt{3}$

Solution:

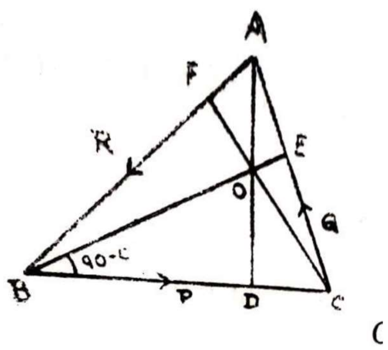


Fig 13

Let AD, BE and CF be the altitudes of the triangle intersecting at O, the orthocentre.
As the resultant passes through O, moment of the resultant about O is zero.
Therefore the sum of the moments about P,Q,R about O is also is zero.
Hence taking moments about O, we have $P.OD+Q.OE+R.OF=0$ ….(1)
In the right angle triangle BOD, $\angle OBD=\angle EBC=90-C \Rightarrow \tan(90-C)=ODBD \Rightarrow OD=BDcotC$ …..(2)
From right angle triangle ABD, $cosB=BDAB \Rightarrow BD=AB.cosB=ccosB$
From (2) $OD=c\cos B.cotC=ccosB.cosCsinC=2R'cosBcosC$

Similarly $OE=2R'cosCcosA$ $OF=2R'cosAcosB$
Hence (1) becomes $P2R'cosCcosA+Q2R'cosCcosA+R2R'cosAcosB=0$
$\Rightarrow PcosA+QcosB+RcosC=0$
Now P,Q,R are being the magnitudes of the forces, are all positive.
Hence in order that in the above relation may hold good, atleast one of the terms must be negative.
Ie) the triangle must be obtuse angled.
Given A=120 and the other angles are equal. Then B=C=30
Therefore the above equation becomes $Pcos120+Qcos30+Rcos30=0$ $\Rightarrow P(-12)+Q+R(\surd 32)=0$
$\Rightarrow P\surd 3=Q+R$

Problem 5:

Forces P,Q,R act along the sides BC,AC,BA respectively of an equilateral triangle. If their resultant is a force parallel to BC through the centroid of the triangle, prove that $Q=R=12P$

Solution:

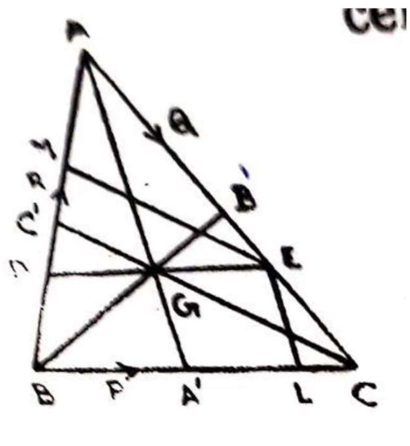
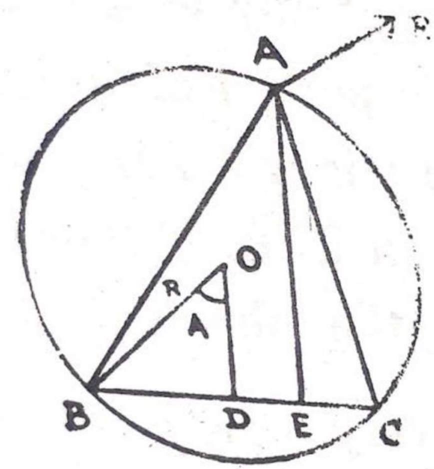


Fig. 14

Given that the triangle ABC is equilateral, the medians AA',BB' and CC' are also the altitudes meeting at G, the centroid.
Let DE be parallel to BC through G.
It is given that DGE is the line of action of the resultant.
As the resultant passes through G, its moment about G is zero.
Therefore sum of the moments of P,Q,R about G is also zero. $\Rightarrow P.GA'-Q.GB'-R.GC'=0$
$\Rightarrow P-Q-R=0$ ….(1)
Since the resultant passes through E also, sum of the moments of P,Q,R about E is zero.
Draw EL perpendicular to BC and EM perpendicular to AB. $\therefore P.EL-R.EM=0$ ….(2)
From the similar triangles ELC and AA'C, $ELAA'=ECAC=13\Rightarrow EL=13AA'$
From the similar triangles AME and AC'C, $EMCC'=AEAC=23\Rightarrow EM=23CC'$
Then the equation (2) becomes $P.13AA'-R23CC'=0$

$\Rightarrow P=2R \Rightarrow R=P2$
Therefore (1) becomes $P-Q-{\scriptstyle P2}=0 \Rightarrow Q=P2$
Hence $Q=R={\scriptstyle P2}$
Problem 6:
A uniform circular plate is supported horizontally at three points A,B,C of its circumference. Show that the pressures on the supports are in the ratio sin2A:sin2B:sin2C
Solution:



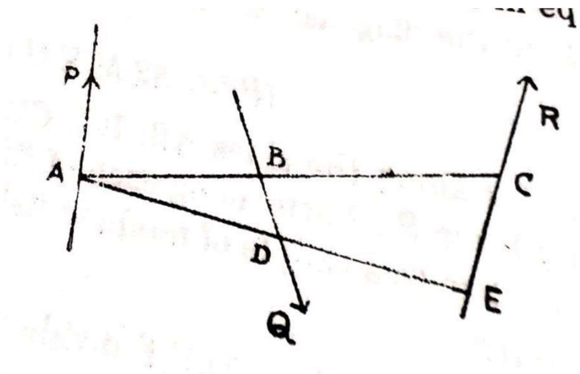
Let BC=a, CA=b, and AB=c. W, the weight of the plate acts at O, the centre of the circle and which is also the circumcentre of the triangle. Let OD be perpendicular to BC. We know that $\angle BOD=A$
From right angle triangle BOD, $OD=OBcos\angle BOD=RcosA$,R being the circumradius of the triangle.
Let AE be perpendicular to BC. $AE=ACsin\angle ACE=bsinC$
Let $R1$ be the reaction at A.
Taking moments about BC, we have $R{\scriptstyle 1}AE=W.OD$

$\Rightarrow R_1 = WRcosAbsinC = WRcosA2RsinBsinC = WcosA2sinBsinC = W2sinAcosA4sinAsinBsinC$
$= Wsin2A4sinAsinBsinC$
Similarly the reactions $R_2\ and\ R_3$ at the other two supports are $Wsin2B4sinAsinBsinC$ and $Wsin2C4sinAsinBsinC$ $\therefore R_1 : R_2 : R_3 = sin2A : sin2B : sin2C$

UNIT - III

EQUILIBRIUM OF THREE FORCES ACTING ON A RIGID BODY

Rigid body subjected to any three forces:



Let P,Q,R be three forces in equilibrium

Take any point A on the line of action of P and any point B on the line of action of Q, such that AB is not parallel to R. Then the three forces being in equilibrium, the sum of their moments about the line AB is zero. But P and Q intersect AB and therefore their moments about AB are each zero.

Hence the moment of R about AB is also zero.

Therefore R is either parallel to AB or R intersects AB.

But we have chosen the points A and B such that R is not parallel to AB

R must intersect AB at a point say C.

Similarly if D is some other point on Q such that AD is parallel to R, we can prove R must intersect AD also at a point say E.

Since the lines BC and De intersect at A, BD and CE must lie in one plane and A is on this plane.

ie A is a point on the plane formed by Q and R.

But A is any point on the line of action of P.

Every point on P is a point on the plane formed by Q and R.

P,q,r are in one plane.

Thus if three forces acting on a rigid body are in equilibrium, they must be coplanar.

Three coplanar forces theorem:

If three coplanar forces acting on a rigid body keep it in equilibrium, they must either be concurrent or be all parallel.

Proof:

Let P,Q,R be three coplanar forces acting on a rigid body and keep it in equilibrium.

Then R must be equal and opposite to the resultant of P and Q.

Now, P and Q being coplanar must either be parallel or intersect.

Case i):

If P and Q are parallel, their resultant is also a parallel force. As R balances the above resultant, it must act in the same line but in opposite direction. So R also is in the same direction as that of P and Q.

i.e) P,Q,R are all parallel to one another.

Case ii)

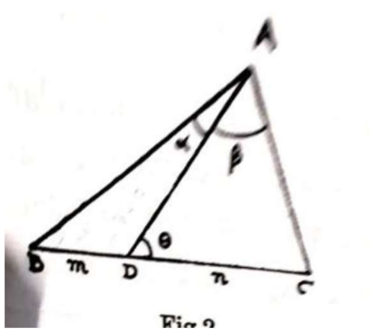
Let P and Q meet at a point O. Then by parallelogram law, their resultant is a force through O. As this is balanced by the third force R, the line of action of R must also pass through O.

i.e) the three forces are concurrent.

Two trigonometrical theorem:

If D is the point on the base BC of the triangle ABC such that $BDDC=mn$ $and\ \angle ADC=\theta,\angle BAD=\alpha$ $\angle DAC=\beta\ then\ (m+n)cot\theta=mcot\alpha-ncot\beta\ and\ (m+n)cot\theta=ncotB-mcotC$

Proof:



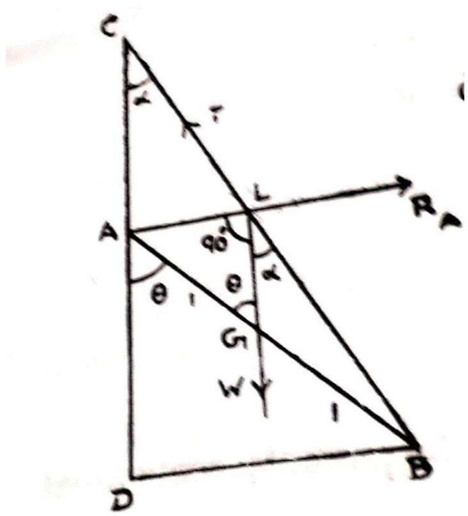
$mn=BDDC$

$=BDDADADC = sin\angle BADsin\angle ABDsin\angle ACDsin\angle DAC = sin\alpha \sin(\theta-\alpha)\sin(\theta+\beta)sin\beta$
$=sin\alpha(sin\theta cos\beta+cos\theta sin\beta)(sin\theta cos\alpha-cos\theta sin\alpha)sin\beta = cot\beta+cot\theta cot\alpha-cot\theta$
$\therefore m(cot\alpha-cot\theta)=n(cot\beta+cot\theta)$ $\therefore(m+n)cot\theta=mcot\alpha-ncot\beta$
Again $mn=sin\angle BADsin\angle ABDsin\angle ACDsin\angle DAC = \sin(\theta-B)sinCsinBsin(C+\theta)$
$=(sin\theta cosB-cos\theta sinB)sinCsinB(sinCcos\theta+cosCsin\theta) = cotB-cot\theta cot\theta+cotC$
$\therefore m(cot\theta+cotC)=n(cotB-cot\theta)$
Hence $(m+n)cot\theta=ncotB-mcotC$

Problem 1:

A uniform rod of length a hangs against a smooth vertical wall being supported by means of a string of length l tied to one end of the rod the other end of the string being attached to a point in the wall. Show that the rod can rest inclined to the wall at an angle θ given by $cos2\theta=l2-a23a2$, what are the limits of the ratio of a:l in order that equilibrium may be possible?

Solution:



AB is the rod of length a, G its centre of gravity and BC is the string of length l. The forces acting on the rod are:

i) Its weight W acting vertically downwards through G.

ii) The reaction R_A at A which is normal to the wall and therefore horizontal.

iii) The tension T of the string along BC.

These three forces in equilibrium not being parallel, must meet in a point L, as shown in the figure. Let the string make an angle α with the vertical $\therefore \angle ACB=\alpha=\angle GLB\ also\ \angle LGB=180-\theta\ and$ $\angle ALG=90$

Using trigonometrical theorem to triangle ALB we have $(1+1)\cot(180-\theta)=1.cot90-1.cot\alpha$
$\Rightarrow -2cot\theta=-cot\alpha \Rightarrow 2cot\theta=cot\alpha$ ...(1)

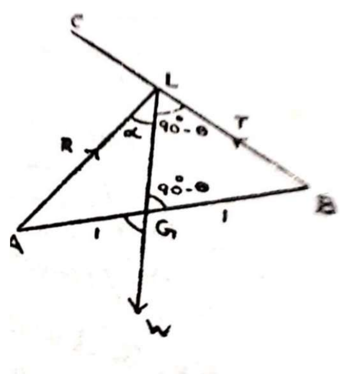
Draw BD perpendicular to CA.

From right angle triangle CDB, $BD=BC\sin\alpha=l\sin\alpha$
And from right angle triangle $BD=AB\sin\theta=a\sin\theta$
Therefore $l\sin\alpha=a\sin\theta$...(2)
Eliminating α between (1) and (2) we use $cosec^2\alpha=1+\cot^2\alpha$ ...(3)
From (2) $\sin\alpha=\frac{a\sin\theta}{l}\Rightarrow cosec\,\alpha=\frac{l}{a\sin\theta}$
Equation (3) becomes $\frac{l^2}{a^2\sin^2\theta}=1+4\cot^2\theta \Rightarrow \frac{l^2}{a^2}=\sin^2\theta+4\cos^2\theta=1+3\cos^2\theta$
$\Rightarrow 3\cos^2\theta=\frac{l^2}{a^2}-1=\frac{l^2-a^2}{a^2} \Rightarrow \cos^2\theta=\frac{l^2-a^2}{3a^2}$
For the above equilibrium position to be possible ,$\cos^2\theta$ must be positive and less than 1.
$l^2-a^2>0\Rightarrow l^2>a^2\Rightarrow a^2<l^2$
Also $\frac{l^2-a^2}{3a^2}<1 \Rightarrow l^2-a^2<3a^2\Rightarrow l^2<4a^2\Rightarrow a^2>\frac{l^2}{4}$
Therefore a^2 lies between $\frac{l^2}{4}$ and l^2 $\frac{a^2}{l^2}$ lies between $\frac{1}{4}$ and 1 $\frac{a}{l}$ lies between $\frac{1}{2}$ and 1.

Problem 2:

A beam of weight W hinged at one end is supported at the other end by a string so that the beam and the string are in a vertical plane and make the same angle θ with the horizon. Show that the reaction at the hinge is $\frac{W}{4}\sqrt{8+cosec^2\theta}$

Solution:



Let AB be the beam of weight W and G its centre of gravity.
BC is the string.
The forces acting on the string are
i) Its weight W acting vertically downwards at G.
ii) The tension T along BC
iii) The reaction R at the hinge at A.

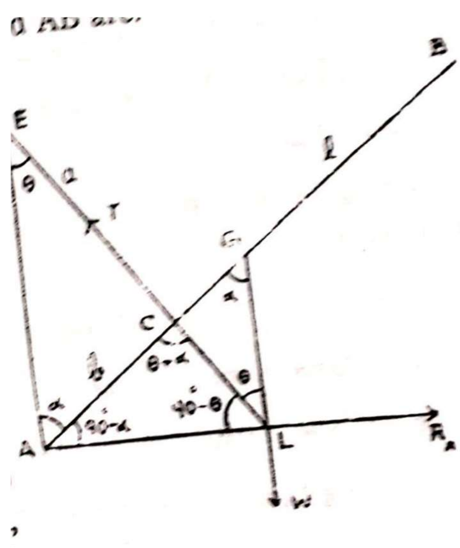
Let the forces (i) and (ii) meet at L.
For equilibrium the third force R must pass through L.
i.e) the reaction at the hinge is a force along AL.
BC and AB makes the same angle θ with the horizon.
They make the same angle 90-θ with the vertical LG.
i.e) $\angle BLG=90-\theta=\angle LGB$
let$\angle ALG=\alpha$
using the triginometrical theorem to triangle ALB, we have
$(1+1)\cot(90-\theta)=1.\cot\alpha-1.\cot(90-\theta) \Rightarrow 2\tan\theta=\cot\alpha-\tan\theta \Rightarrow 3\tan\theta=\cot\alpha$
Applying lami's theorem for the three forces at L, we have

$R\sin(90-\theta)=W\sin(90-\theta+\alpha) \Rightarrow R cos\theta=W\cos(\theta-\alpha) \Rightarrow R=W cos\theta\cos(\theta-\alpha)$
$=W cos\theta\cos\theta cos\alpha+sin\theta sin\alpha$ $=W cos\theta\sin\alpha(\cos\theta\cot\alpha+\sin\theta)$ $=W cos\theta\sin\alpha(\cos\theta 3\tan\theta+\sin\theta)$
$=W cos\theta\sin\alpha(3\sin\theta+\sin\theta)$ $=W cos\theta\sin\alpha 4\sin\theta$ $=W 4 cot\theta\sqrt{}1+cot2\alpha$ $=W 4 cot\theta\sqrt{}1+9 tan2\theta$
$=W 4\sqrt{}cot2\theta+9$ $=W 4\sqrt{}cosec2\theta+8$

Problem3:

A uniform rod of length 2l rests with its lower end in contact with a smooth vertical wall. It is supported by a string of length a, one end of which is fastened to a point in the wall and the other end to a point in the rod at a distance b from its lower end. If the inclination of string to the vertical wall be θ, show that $cos2\theta=b2(a2-b2)a2(2b-l)$

Solution:



The forces acting on the rod are

i) The reaction at A perpendicular to the wall and hence horizontal

ii) Its weight W acting vertically downwards through G, the midpoint of Ab

iii) The tension T of the string along CE.

For equilibrium the three forces must be meet at one point sayL

From triangle ACE $ACsin\theta=ECsin\alpha \Rightarrow bsin\theta=asin\alpha \Rightarrow bsin\alpha=asin\theta$ ...(1)

From triangle ACL, $CL\sin(90-\alpha)=AC\sin(90-\theta) \Rightarrow CLcos\alpha=bcos\theta \Rightarrow CL=bcos\alpha cos\theta$ ...(2)

From triangle CGL, $CLsin\alpha=CGsin\theta \Rightarrow CL=CGsin\alpha sin\theta=(AG-AC)sin\alpha sin\theta=(l-b)sin\alpha sin\theta$ ...(3)

Equating (2) and (3) we get $bcos\alpha cos\theta=(l-b)sin\alpha sin\theta$

$\Rightarrow cot\alpha=(l-b)bcot\theta$

We know that $cosec2\alpha=1+cot2\alpha \Rightarrow b2a2sin2\theta=1+(l-b)2cot2\theta b2 \Rightarrow b2a2=sin2\theta+(l-b)2cos2\theta b2$

$=1-cos2\theta+(l-b)2cos2\theta b2 =1-cos2\theta(1-(l-b)2b2) =1-cos2\theta(b2-l2-b2+2blb2)$

$=1-cos2\theta(l(2b-l)b2) \Rightarrow cos2\theta(l(2b-l)b2)=1-b2a2 \Rightarrow cos2\theta=b2(a2-b2)a2(2b-l)$

UNIT – IV

FRICTION

Definition:

1. If two bodies are in contact with one another, the property of the two bodies, by means of which a force is exerted between them at their point of contact to prevent one body from sliding on the other, is called the friction. The force exerted is called the force of friction.

2. When one body in contact with another is in equilibrium, the friction exerted is just sufficient to maintain equilibrium and is called statical friction.

3. When one body is just on the point of sliding on another, the friction exerted attains its maximum value and is called limiting friction, the equilibrium in this case is said to be limiting

4. When motio ensues by one body sliding over another, the friction exerted is called dynamical friction.

Laws of friction:

1. When two bodies are in contact, the direction of friction on one of them at the point of contact is opposite to the direction in which the point of contact would commence to move.

2. When there is equilibrium, the magnitude of friction is just sufficient to prevent the body from moving.

3. The magnitude of the limiting friction always bears a constant ratio to the normal reaction and this ratio depends only on the substances of which the bodies are compossed.

4. The limiting friction is independent of the extent and shape of the surfaces in contact, so long as the normal reaction is unaltered.

5. When motion ensures by one body sliding over the other, the direction of friction is opposite to that of motion; the magnitude of the friction is independent of the velocity of the point of contact but the ratio of the friction to the normal reaction is slightly less when the body moves, than when it is in limiting equilibrium.

Coefficient of friction:

The ratio of the limiting friction to the normal reaction is called the coefficient of friction and it is denoted by μ.
Let F be the friction and R be the normal reaction, then $FR=\mu \Rightarrow F=\mu R$

Angle of friction:

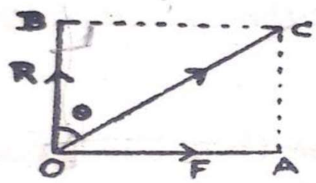


Fig. 1

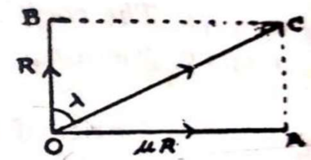


Fig. 2

Suppose one body is kept in equilibrium by friction on another. At the point of contact Q, two forces act on the first body, namely the normal reaction and the force of friction, these two act in perpendicular directions and they can be compounded into a single force. This single force is called the resultant reaction or the total reaction.

In diagram. let $\overline{OA}=F$, the forces of friction and $\overline{OB}=R$ the normal reaction. Let $\overline{OC}$ be the resultant of F and R.

If $\angle BOC=\theta, tan\theta=BCOB=OAOB=FR$

As F increases , the value θ increases until the friction F reaches ita maximum value. In that case, equilibrium is limiting and the angle made by the resultant reaction with the normal is called the angle of friction.

Hence the greatest value of θ is λ

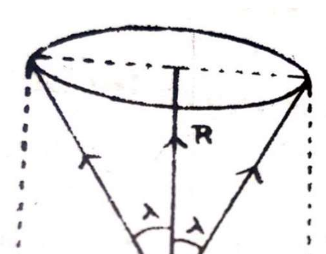
When one body is in limiting equilibrium over another, the angle which the resultant reaction makes with the normal at the point of contact is called the angle of friction and is denoted by λ.

In the second figure $\overline{OA}$ represent the limiting friction which is equal to μR, μ being the coefficient of friction.

$\overline{OC}$is the resultant of $\overline{OA}$ and $\overline{OB}$ $\angle BOC=\lambda=angle\ of\ friction.\ tan\lambda=BCOB=OAOB=\mu RR=\mu$

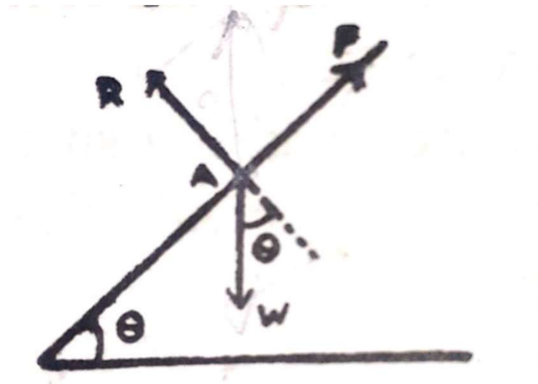
Thus the coefficient of friction is equal to the tangent of the angle of friction.

Cone of friction:



When two bodies are in contact, we can consider a cone drawn with the point of contact as the vertex, the common normal as the axis and its semi vertical angle being equal to λ, the angle of friction. Such a cone is called the cone of friction.

Equilibrium of a particle on a rough inclined plane



a particle of weight W be placed at A on a rough inclined plane, whose inclination to the horizon is θ. The forces acting on it are

1. Its weight W acting vertically downwards.
2. The frictional force F acting along the inclined plane upwards.
3. The normal reaction R perpendicular to the plane.

Resolving along and perpendicular to the plane, we get $F=Wsin\theta$ ...(1) $R=Wcos\theta$ ....(2)
$\therefore FR=tan\theta$

We know that $\frac{F}{R}$ is always less than μ
Hence for equilibrium $\tan\theta<\mu$
i.e) $\tan\theta<\tan\lambda$, λ being the angle of friction
suppose θ, the inclination of the plane is gradually increased
when θ=λ, then $\frac{F}{R}=\tan\lambda=\mu$
in this case the equilibrium becomes limiting and the particle is just on the point of sliding down.
Hence if a body be placed on a rough inclined plane and be on the point of sliding down the plane under the action of its weight and the reaction of the plane only, the angle of inclination of the plane to the horizon is equal to the angle of friction.

Equilibrium of a body on a rough inclined plane under a force parallel to the plane:

Theorem:A body is at rest on a rough plane inclined to the horizon at an angle greater than the angle of friction and is acted upon by a force, parallel to the plane and along the line of greatest slope; to find the limits between which the force must lie.

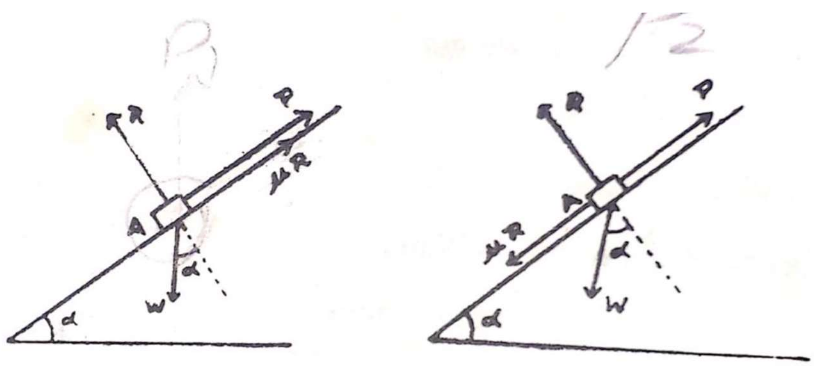


Fig. 5

Let αbe the inclination of the plane to the horizon. W the weight of the body and R the normal reaction.
Case i) Let the body be on the point of moving down the plane. The limiting friction acts up the plane and is equal toμR. Let P be the force required to keep the body at rest.
Resolving along and perpendicular to the plane, we have $P+\mu R=W\sin\alpha$ …(1) $R=W\cos\alpha$ …(2)

(1) Becomes $P+\mu W cos\alpha = W sin\alpha$

If λ is the angle of friction ,$\mu = tan\lambda$ $\therefore P = W sin\alpha - tan\lambda W cos\alpha = W(sin\alpha - sin\lambda cos\lambda cos\alpha)$
$= W sin(\alpha - \lambda) cos\lambda$
Case ii) Let the body be on the point of moving up the plane. The limiting friction acts down the plane and is equal toμR. Let P be the force required to keep the body at rest.
Resolving along and perpendicular to the plane, we have $P - \mu R = W sin\alpha$ ...(1) $R = W cos\alpha$ ...(2)
(2) Becomes $P - \mu W cos\alpha = W sin\alpha$

If λ is the angle of friction ,$\mu = tan\lambda$ $\therefore P = W sin\alpha + tan\lambda W cos\alpha = W(sin\alpha + sin\lambda cos\lambda cos\alpha)$
$= W sin(\alpha + \lambda) cos\lambda$
Hence the equilibrium of the force P must lie between the values $W sin(\alpha - \lambda) cos\lambda$ and $W sin(\alpha + \lambda) cos\lambda$
Equilibrium of a body on a rough inclined plane under any force
Theorem: A body is at rest on a rough inclined plane of inclination α to the horizon, being acted on by a force making an angle θ with the plane; to find the limits between which the force must lie and also to find the magnitude and direction of the least force required to drag the body up the inclined plane

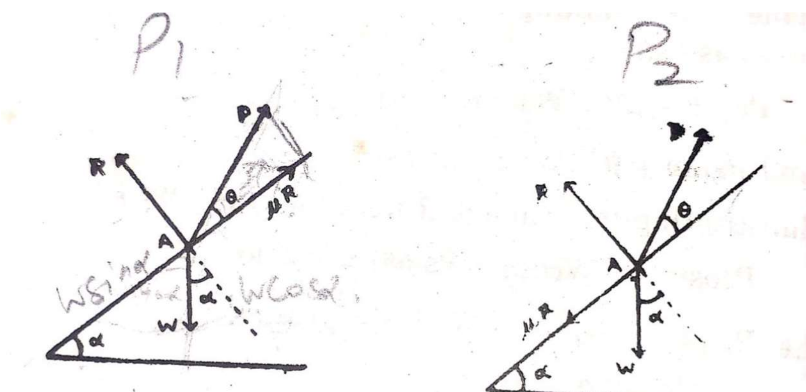


Fig. 7

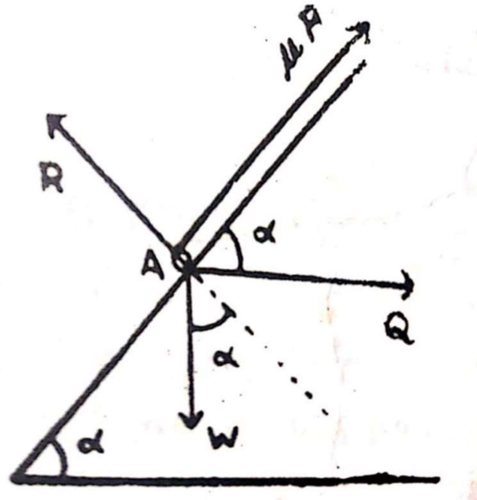
Let W be the weight of the body, P the force acting at an angle θ with the plane and R the normal reaction.
Casei) Let the body in just on the point of moving down the plane. Then the limiting friction μR acts upwards. Resolving the forces along and perpendicular to the plane, we get $P cos\theta + \mu R = W sin\alpha$
...(1) $P sin\theta + R = W cos\alpha$ ...(2) $\Rightarrow R = W cos\alpha - P sin\theta$ $\therefore$(1)$\Rightarrow P cos\theta + \mu(W cos\alpha - P sin\theta) = W sin\alpha$
$\Rightarrow P(cos\theta - \mu sin\theta) = W(sin\alpha - \mu cos\alpha)$ $\Rightarrow P = W(sin\alpha - \mu cos\alpha)(cos\theta - \mu sin\theta)$
If λ is the angle of friction $\mu = tan\lambda$
Then $P = W(sin\alpha - tan\lambda cos\alpha)(cos\theta - tan\lambda sin\theta) = W(cos\lambda sin\alpha - sin\lambda cos\alpha)(cos\lambda cos\theta - sin\lambda sin\theta)$
$= W sin(\alpha - \lambda) cos(\theta + \lambda)$
Caseii) Let the body in just on the point of moving up the plane. Then the limiting friction μR acts downwards. Resolving the forces along and perpendicular to the plane, we get

$P\cos\theta - \mu R = W\sin\alpha$ ...(1) $P\sin\theta + R = W\cos\alpha$ ...(2) $\Rightarrow R = W\cos\alpha - P\sin\theta$
$\therefore (1) \Rightarrow P\cos\theta - \mu(W\cos\alpha - P\sin\theta) = W\sin\alpha \Rightarrow P(\cos\theta + \mu\sin\theta) = W(\sin\alpha + \mu\cos\alpha)$
$\Rightarrow P = W(\sin\alpha + \mu\cos\alpha)(\cos\theta + \mu\sin\theta)$
If λ is the angle of friction $\mu = \tan\lambda$
Then $P = W(\sin\alpha + \tan\lambda\cos\alpha)(\cos\theta + \tan\lambda\sin\theta) = W(\cos\lambda\sin\alpha + \sin\lambda\cos\alpha)(\cos\lambda\cos\theta + \sin\lambda\sin\theta)$
$= W\sin(\alpha+\lambda)\cos(\theta-\lambda)$
Hence if body lies between $W\sin(\alpha-\lambda)\cos(\theta+\lambda)$ and $W\sin(\alpha+\lambda)\cos(\theta-\lambda)$, the body will remain in equilibrium.

Problem 1:

A weight can be supported on a rough inclined by a force P acting along the plane or by a force Q acting horizontally. Show that the weight is $PQ\sqrt{Q^2\sec^2\lambda - P^2}$ where λ is the angle of friction.

Solution:



Let W be the weight and α be the angle of inclination of the plane. R is the normal reaction.
When the weight is just on the point of moving down, limiting friction μR acts upwards. A horizontal force Q keeps the weight in equilibrium.
Resolving along and perpendicular to the plane, $\mu R + Q\cos\alpha = W\sin\alpha$ ...(1)
And $R = W\cos\alpha + Q\sin\alpha$ ...(2)
(1) Becomes $\mu(W\cos\alpha + Q\sin\alpha) + Q\cos\alpha = W\sin\alpha$

$\Rightarrow Q(\mu\sin\alpha + \cos\alpha) = W(\sin\alpha - \mu\cos\alpha) \Rightarrow \cos\alpha(Q+\mu W) = \sin\alpha(W-\mu Q)$
$\Rightarrow \cos\alpha(W-\mu Q) = \sin\alpha(Q+\mu W)$ *and each* $= \sqrt{\cos^2\alpha + \sin^2\alpha}\sqrt{(W-\mu Q)^2 + (Q+\mu W)^2}$
$= 1\sqrt{\mu^2Q^2 + W^2 - 2\mu QW + Q^2 + \mu^2W^2 + 2\mu QW} = 1\sqrt{Q^2(1+\mu^2) + W^2(1+\mu^2)} = 1\sqrt{1+\mu^2}\sqrt{Q^2+W^2}$
$= 1\sqrt{1+\tan^2\lambda}\sqrt{Q^2+W^2} = 1\sec\lambda\sqrt{Q^2+W^2}$

therefore $\cos\alpha=\frac{(W-\mu Q)\sec\lambda}{\sqrt{Q^2+W^2}}$ *and* $\sin\alpha=\frac{(Q+\mu W)\sec\lambda}{\sqrt{Q^2+W^2}}$

The same weight W is supported by a force P acting along the plane.

Then $P=\frac{W\sin(\alpha-\lambda)}{\cos\lambda}=\frac{W}{\cos\lambda}(\sin\alpha\cos\lambda-\cos\alpha\sin\lambda)$

$=\frac{W}{\cos\lambda}\left(\frac{(Q+\mu W)\sec\lambda}{\sqrt{Q^2+W^2}}\cos\lambda-\frac{(W-\mu Q)\sec\lambda}{\sqrt{Q^2+W^2}}\sin\lambda\right)=\frac{W}{\sqrt{Q^2+W^2}}(Q(\cos\lambda+\mu\sin\lambda)+W(\mu\cos\lambda-\sin\lambda))=\frac{W}{\sqrt{Q^2+W^2}}(Q(\cos\lambda+\tan\lambda\sin\lambda)+W(\tan\lambda\cos\lambda-\sin\lambda))=\frac{W}{\sqrt{Q^2+W^2}}(Q\cos\lambda)=\frac{WQ\sec\lambda}{\sqrt{Q^2+W^2}}$

Then $P^2=\left(\frac{WQ\sec\lambda}{\sqrt{Q^2+W^2}}\right)^2=\frac{W^2Q^2\sec^2\lambda}{(Q^2+W^2)}\Rightarrow P^2(Q^2+W^2)=W^2Q^2\sec^2\lambda$

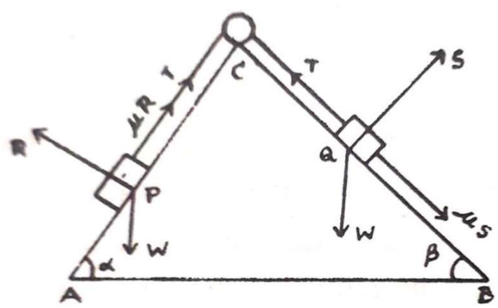
$\Rightarrow W^2(-P^2+Q^2\sec^2\lambda)=P^2Q^2$

$\Rightarrow W=\frac{PQ}{\sqrt{Q^2\sec^2\lambda-P^2}}$

Problem 2:

Two particles P and Q each of weight W on two equally rough inclined planes CA and CB of the same height, placed back to back are connected by a light string which passes over the smooth top edge C of the planes. Show that if particles are on the point of slipping, the difference of the inclination of the plane is double the angle of friction.

Solution:



Let α and β be the inclination of the planes CA and CB: R,S be the normal reactions of the planes, T the tension of the string and μthe coefficient of friction.

Let P be the point of moving downwards. Then Q will be moving upwards.

Limiting friction μR will act on P upwards the inclined plane and the limiting friction μS will act on Q downwards the inclined plane.

Considering the equilibrium of P and resolving along and perpendicular to the plane CA, we have

$\mu R+T=W\sin\alpha$ …(1) $R=W\cos\alpha$ …(2)

(1) $\Rightarrow\mu(W\cos\alpha)+T=W\sin\alpha$

$\Rightarrow T=W\sin\alpha-\mu W\cos\alpha$ …(3) *resolving along and perpendicular to the plane CB, we have*
$T=W\sin\beta+\mu S$ …(4) $S=W\cos\beta$ …(5) ∴(4)$\Rightarrow T=W\sin\beta+\mu W\cos\beta$ …(6)

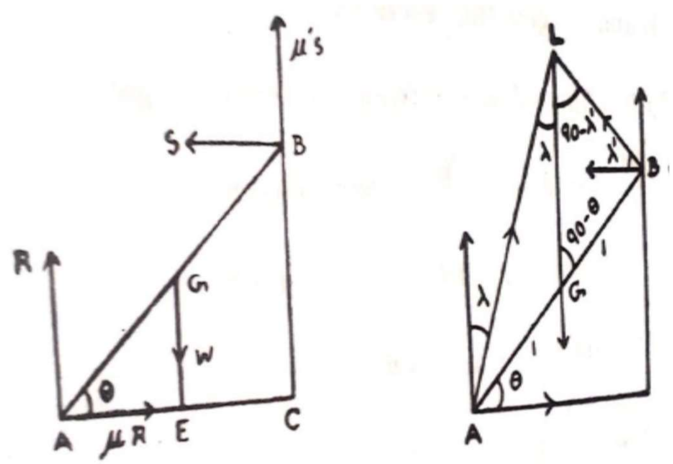
Equating the two values of T we get, $W\sin\alpha-\mu W\cos\alpha=W\sin\beta+\mu W\cos\beta$

$\Rightarrow\mu(\cos\beta+\cos\alpha)=\sin\alpha-\sin\beta$

$\Rightarrow\mu=\frac{\sin\alpha-\sin\beta}{(\cos\beta+\cos\alpha)}=\frac{2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}=\tan\frac{\alpha-\beta}{2}$

If λ is the angle of friction, $\mu=\tan\lambda=\tan\frac{\alpha-\beta}{2}\Rightarrow\lambda=\frac{\alpha-\beta}{2}\Rightarrow 2\lambda=\alpha-\beta$

Problem 3:
A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall;if the ground and wall be both rough, the coefficient of friction being μ and μ' respectively, and if the ladder be on the point of slipping at both ends, show that θ, the inclination of the ladder to the horizon is given by $tan\theta=\frac{1-\mu\mu'}{2\mu}$. Find also the reactions at the wall and ground.
Solution:



Let AB be the ladder, G its centre of gravity and W its weight. Let R and S be the normal reactions acting on the ladder at the ground and wall.
Resolving horizontally, $S=\mu R$ ...(1)
Resolving vertically, $\mu' S+R=W$ ....(2)

$\Rightarrow \mu' \mu R + R = W \Rightarrow R = W 1 + \mu \mu,$

Equation (1) becomes $S = \mu W 1 + \mu \mu,$

Taking moments about A, $S.BC + \mu' S.AC = W.AE \Rightarrow S.2 a sin\theta + \mu' S 2 a sin\theta = W a cos\theta$ ...(3)

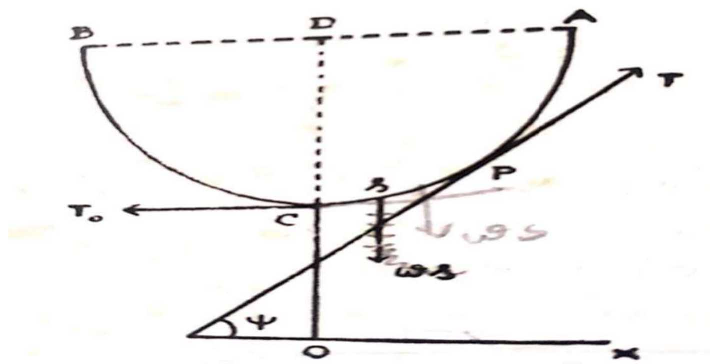
$\Rightarrow \mu W 1 + \mu \mu, 2 sin\theta + \mu \mu, W 1 + \mu \mu, 2 cos\theta = W cos\theta \Rightarrow 2\mu \sin\theta + 2\mu\mu' \cos\theta = (1 + \mu\mu') \cos\theta$

$\Rightarrow 2\mu \sin\theta = -2\mu\mu' \cos\theta + (1 + \mu\mu') \cos\theta \Rightarrow 2\mu \sin\theta = (1 - \mu\mu') \cos\theta$ $tan\theta = 1 - \mu\mu' 2\mu$

UNIT - V

EQUILIBRIUM OF STRINGS

Definition:
If the weight per unit length of the chain or string is constant, the catenary is called the uniform or common catenary.
Equation of the common catenary:



Let ACB be a uniform heavy flexible card attached to two points A and B at the same level, C being the lowest of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the arc CP=s.
Let w be the weight per unit length of the chain.
Consider the equilibrium of the portion CP of the chain.
The forces acting on it are:
1. Tension T_0 acting along the tangent at C and which is therefore horizontal.
2. Tension T acting at P along the tangent at P making an angle ψ with OX.
3. Its weight ws acting vertically downwards through the centre of gravity of the arc CP.

For equilibrium, these three forces are must be concurrent.
Hence the line of action of the weight we must pass through the point of intersection of T and T_0
Resolving horizontally and vertically, we have $T cos\psi = T_0$ ...(1)

$T\sin\psi = ws$ ...(2) $\Rightarrow \tan\psi = \frac{ws}{T_0}$

We shall write $T_0 = wc$ *where* c is constant. $\therefore \tan\psi = \frac{s}{c}$

$\Rightarrow s = c\tan\psi$ which is called the intrinsic equation of the catenary.

to obtain Cartesian equation of the common catenary:

we know that the relation $\frac{dy}{ds} = \sin\psi$ and $\frac{dy}{dx} = \tan\psi$

now $\frac{dy}{d\psi} = \frac{dy}{ds}.\frac{ds}{d\psi} = \sin\psi c\sec^2\psi = c\sec\psi\tan\psi \Rightarrow y = \int c\sec\psi\tan\psi d\psi = c\sec\psi + A$

If y=c when ψ=0, then c=sec0+A

Therefore A=0. $y = c\sec\psi$ $y^2 = c^2\sec^2\psi = c^2(1+\tan^2\psi) = c^2 + s^2$ $\frac{dy}{dx} = \tan\psi = \frac{s}{c} = \frac{\sqrt{y^2 - c^2}}{c}$

$\Rightarrow \frac{dy}{\sqrt{y^2 - c^2}} = \frac{dx}{c}$

Integrating $\cosh^{-1}\frac{y}{c} = \frac{x}{c} + B$

When x=0, y=c

Therefore B=0.

Hence $\cosh^{-1}\frac{y}{c} = \frac{x}{c} \Rightarrow y = c\cosh\frac{x}{c}$

The above equation is the Cartesian equation of the common catenary.

Tension at any point:

Let ACB be a uniform heavy flexible card attached to two points A and B at the same level, C being the lowest of the cord. Draw CO vertical, OX horizontal and take OX as X axis and OC as Y axis. Let P be any point of the string so that the length of the arc CP=s.
Let w be the weight per unit length of the chain.
Consider the equilibrium of the portion CP of the chain.
The forces acting on it are:

1. Tension T_0 acting along the tangent at C and which is therefore horizontal.
2. Tension T acting at P along the tangent at P making an angle ψ with OX.
3. Its weight ws acting vertically downwards through the centre of gravity of the arc CP.

For equilibrium, these three forces are must be concurrent.
Hence the line of action of the weight we must pass through the point of intersection of T and T_0
Resolving horizontally and vertically, we have $T\cos\psi = T_0$ ...(1) $T\sin\psi = ws$ ...(2)
Squaring (1) and (2) and adding we get $T^2 = T_0^2 + w^2s^2 = w^2c^2 + w^2s^2 = w^2(c^2+s^2) = w^2y^2$
Therefore T=wy

Problem 1:

A uniform chain of length l is to be suspended from two points in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span must be
$\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$

Solution:

Let y_A and y_C be the y-coordinates of the highest point A and the lowest point C. Let w be the weight per unit length of the chain and c the parameter of the catenary.
Tension at A is wy_A
Tension at C is wy_C
Now $wy_A = nwy_C$

$\Rightarrow y_A = n y_C = nc \Rightarrow c\cosh\frac{x_A}{c} = nc \Rightarrow \cosh\frac{x_A}{c} = n \Rightarrow x_A = c\cosh^{-1}n = \log(n+\sqrt{n^2-1})$

We have to find c.

$y_A^2 = c^2 + s_A^2$, s_A denoting the length of CA. $y_A^2 = c^2 + \frac{l^2}{4} \Rightarrow n^2c^2 = c^2 + \frac{l^2}{4} \Rightarrow c^2(n^2-1) = \frac{l^2}{4}$

$\Rightarrow c^2 = \frac{l^2}{4(n^2-1)} \Rightarrow c = \frac{l}{2\sqrt{n^2-1}}$

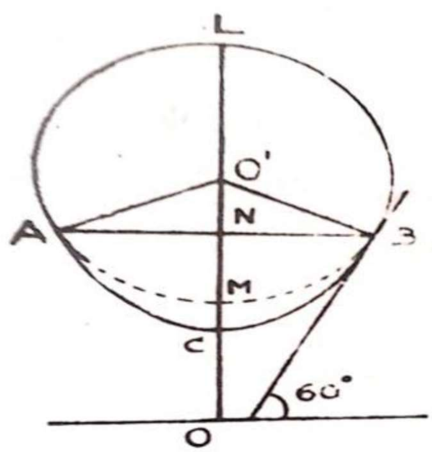
Hence $x_A = \frac{l}{2\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$

Span AB= $\frac{l}{\sqrt{n^2-1}}\log(n+\sqrt{n^2-1})$

Problem 2:

Shoe that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two thirds of the circumference of the pulley is $a\left[\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right]$

Solution:



Let CBLAC be an endless chain hanging over the circular pulley MBLA of radius a.

The portion ALB=two third of the circumference of the pulley= $\frac{2}{3}2\pi a = \frac{4\pi a}{3}$

The remaining portion ACB will hang in the form of the catenary with C as the lowest point.

The tangent at B is perpendicular to O'B and so it makes an angle 60 to the horizontal.

Let the origin O,as usual be taken at a depth c below C. B is the point on the circle and the catenary.

X coordinates of B=NB=O'Bcos30= $\frac{a\sqrt{3}}{2}$

Since B is also on the catenary, $x = c\log(\sec\psi + \tan\psi)$

Applying in the point of B, we have $\psi = 60$, we have $\frac{a\sqrt{3}}{2} = c\log(\sec 60 + \tan 60) = c\log(2+\sqrt{3})$

$\therefore c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$

Now $s = c\tan\psi = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}\tan 60 = \frac{a\sqrt{3}.\sqrt{3}}{2\log(2+\sqrt{3})} = \frac{3a}{2\log(2+\sqrt{3})}$

Hence the length of the chain= $\frac{4\pi a}{3} + \frac{3a}{2\log(2+\sqrt{3})} = a\left[\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right]$